

φ -CANTORIAN FUNCTIONS AND THEIR CONVEX MODULI

E. M. BEESLEY AND A. P. MORSE

Mr. Beesley's contribution to this paper is Part II of a thesis
submitted by him in partial fulfillment of the requirements
for the degree of Doctor of Philosophy at Brown University
June 1943

Reprinted from DUKE MATHEMATICAL JOURNAL
Vol. 12, No. 4, December, 1945

φ -CANTORIAN FUNCTIONS AND THEIR CONVEX MODULI

BY E. M. BEESLEY AND A. P. MORSE

1. Introduction. If φ belongs to the class P (see 3.1), then there is an associated set function φ^* (see 3.2). The possibility of such an association has been noted by Hausdorff [7] and Hahn [6; 450]. This function φ^* is a regular measure in the sense of Carathéodory. We examine several properties of this measure (see §3). We then turn our attention to a class of families of intervals. Each family is required to satisfy a rather simple condition with regard to a function φ and a closed interval of which each member of the family is a subset. We consider the φ^* -measure of sets which are expressible as intersections of sequences of point set sums of intervals of a family. The classical example of Cantor and other similar constructions are included as well as the results of less regular procedures. The sets which we investigate here have φ^* -measure which is neither zero nor infinite. We might point out that in most of the specializations here considered, each set whose Lebesgue measure is finite has infinite φ^* -measure; in particular, the measure of an interval may be infinite.

We next add the requirement that the basic function shall be convex upward. This does not alter the measure theory involved but does allow us to set up certain rather regular procedures for construction of sets (see §5). In some instances this allows us to correlate our work with that of earlier writers whose viewpoint was different from ours. In particular, by our methods some results of Hausdorff [7] with regard to the dimension of sets can be obtained.

After establishing a basis for constructions, we introduce the notion of φ -Cantorian sets associated with a convex function φ and of a φ -Cantorian function associated with each of these sets (see §6). Our theory makes possible the precise determination of what we call the convex modulus for numerous singular functions including the classical one of Cantor.

We employ our theory to demonstrate the possibility of constructing a set of Lebesgue measure zero which cannot be covered in a prescribed manner. The solution to this problem is a φ -Cantorian set associated with a convex function φ whose construction we give (see §7). We are indebted to F. Smithies for a suggestion in this connection.

Construction Theorem 8.2 is, it is felt, sufficiently flexible to be of considerable use in concoction of examples and counter-examples. The relations between results obtainable from this theorem and known results concerning the Hausdorff

Received August 24, 1944; revision received September 10, 1945. Presented to the American Mathematical Society November 24, 1945. The authors wish to acknowledge the assistance of the Research Committee of the University of Nevada. Mr. Beesley's contribution to this paper is submitted as Part II of a thesis in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Brown University.

dimension of certain sets and the Lipschitz conditions satisfied by certain singular functions are discussed (see 8.3).

In the final section we exhibit a perfect plane set and certain affine transformations thereof which casts light upon the theory of product measure and the limitations to the principles of Fubini and Cavalieri. This set is defined simply in terms of decimal representation of coordinates of its elements and has congruent projections on the axes. In defining the product measure, we use the same measure on each axis. As initially defined, this set has product measure one, but after a shear or a rotation, the transformed set is inexpressible as a sum of countably many sets of finite measure.

2. Definitions and notation.

2.1. $R = E_x[-\infty < x < \infty]$.

2.2 If $x \in R$, then by $|x|$ we mean the absolute value of x . If $A \subset R$, then by $|A|$ we mean the Lebesgue outer measure of A .

2.3. $[a, b] = E_x[a \leq x \leq b]$, $(a, b) = E_x[a < x < b]$,

$(a, b) = E_x[a < x < b]$, $(a, b] = E_x[a < x \leq b]$.

2.4. J is a *closed interval* if and only if $J = [a, b]$ with $-\infty < a < b < \infty$.

J is an *open interval* if and only if $J = (a, b)$ with $-\infty < a < b < \infty$.

J is an *interval* if and only if J is an open interval or J is a closed interval.

2.5. $\{x\} = E_y[y = x]$. This notation will be used only in this sense.

2.6. If H is a family and if A_x is a set for $x \in H$, then

$$\sum_{x \in H} A_x = E_y[y \in A_x \text{ for some } x].$$

2.7. If H is a family, then $\sigma(H) = \sum_{x \in H} x$. That is, $y \in \sigma(H)$ if and only if there is a set $x \in H$ such that $y \in x$.

2.8. If H is a countable family and $a_x \geq 0$ for $x \in H$, then

$$\sum_{x \in H} a_x$$

represents the appropriate numerical sum.

2.9. A family H covers A or is a covering of A if and only if $A \subset \sigma(H)$.

2.10. If H is a family of intervals, then $\text{norm } H = \sup_{J \in H} |J|$.

2.11. If $A \subset R$, then $A^\square = R - A$.

2.12. By distance between A and B we mean the inf of numbers of the form $|x - y|$ where $x \in A$, $y \in B$.

3. Properties of the measure.

3.1. DEFINITION. We say that $\varphi \in P$ if and only if φ is a non-decreasing function on $[0, \infty)$ which is continuous at the origin with $\varphi(0) = 0$.

3.2. DEFINITION. If $\varphi \in P$, $S \subset R$, and α is a positive number, we define $\varphi_\alpha^*(S)$ to be the inf of norms of the form

$$\sum_{J \in H} \varphi(|J|),$$

where H is a countable family of closed intervals with norm $H \leq \alpha$ and $S \subset \sigma(H)$.

3.3. DEFINITION.

$$\varphi^*(S) = \lim_{\alpha \rightarrow 0} \varphi_\alpha^*(S).$$

Since $\alpha < \beta$ implies $\varphi_\beta^*(S) \leq \varphi_\alpha^*(S)$, this limit must exist finite or infinite. One can show that the six following theorems hold.

3.4. THEOREM. If $S \subset R$, then $\varphi^*(S)$ is plus infinity or a non-negative real number.

3.5. THEOREM. If S is vacuous, then $\varphi^*(S) = 0$.

3.6. THEOREM. If $A \subset B \subset R$, then $\varphi^*(A) \leq \varphi^*(B)$.

3.7. THEOREM. If F is a countable family of subsets of R , then

$$\varphi^*(\sigma(F)) \leq \sum_{S \in F} \varphi^*(S).$$

3.8. THEOREM. If A and B are subsets of R with positive distance between them, then $\varphi^*(A) + \varphi^*(B) = \varphi^*(A + B)$.

3.9. THEOREM. If $S \subset R$, then there is a set $S' \supset S$ such that S' is the intersection of a countable number of open sets and $\varphi^*(S) = \varphi^*(S')$.

3.9 is a consequence of the following theorem.

3.10. THEOREM. If $\varphi \in P$, $A \subset R$, and $\epsilon > 0$, then there is a countable family H of open intervals such that $A \subset \sigma(H)$, norm $H < \epsilon$ and

$$\sum_{J \in H} \varphi(|J|) \leq \varphi^*(A) + \epsilon.$$

Proof. Let H_1 be a countable family of closed intervals such that $A \subset \sigma(H_1)$, norm $H_1 < \epsilon$ and

$$\sum_{J \in H_1} \varphi(|J|) \leq \varphi^*(A) + \frac{\epsilon}{2}.$$

Let

$$H_2 = \sum_{J \in H_1} \{\text{interior of } J\},$$

$$A_1 = A - \sigma(H_2).$$

Then A_1 is a countable set and since $\varphi \in P$, there is a countable family H_3 of open intervals with $A_1 \subset \sigma(H_3)$, $\text{norm } H_3 < \epsilon$ and

$$\sum_{J \in H_3} \varphi(|J|) < \frac{\epsilon}{2}.$$

Taking $H = H_2 + H_3$ we see that H is a countable family of open intervals with $A \subset \sigma(H)$, $\text{norm } H < \epsilon$ and

$$\begin{aligned} \sum_{J \in H} \varphi(|J|) &\leq \sum_{J \in H_2} \varphi(|J|) + \sum_{J \in H_3} \varphi(|J|) \\ &\leq \varphi^*(A) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \varphi^*(A) + \epsilon. \end{aligned}$$

In view of 3.4, 3.5, 3.6, 3.7, and 3.8, φ^* is a measure in the sense of Carathéodory [3], [4; Chapter V], [6], [9; Chapter II]. These five theorems correspond to conditions of Carathéodory. According to his general theory [3], [4], [6], [9], if ψ is a measure, then $S \subset R$ is said to be ψ -measurable if and only if

$$\psi(T) = \psi(T \cdot S) + \psi(T \cdot S^c)$$

for $T \subset R$. We have, therefore, the following theorem.

3.11. THEOREM. S is φ^* -measurable if and only if $S \subset R$ and

$$\varphi^*(T) = \varphi^*(T \cdot S) + \varphi^*(T \cdot S^c) \text{ for } T \subset R.$$

Also from the general theory it follows that the class of φ^* -measurable sets includes all Borel sets.

A Carathéodory measure ψ is said to be regular if for each set $S \subset R$ there is a set $S' \subset R$ which is ψ -measurable with $S \subset S'$ and $\psi(S) = \psi(S')$. The sixth theorem (3.9) implies that φ^* is a regular measure. More precisely, 3.9 corresponds exactly to a condition of Hahn [6; 444] and [9; 50] which demands somewhat more than regularity.

We also call attention to the following easily proved theorems.

3.12. THEOREM. If $A \subset R$ is a countable set, then $\varphi^*(A) = 0$.

3.13. THEOREM. If $A \subset R$ and $B \subset R$ are congruent sets, then

$$\varphi^*(A) = \varphi^*(B).$$

3.14. LEMMA. If $\varphi^*(A) < \infty$ and

$$\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} < \infty,$$

then

$$|A| \leq \varphi^*(A) \limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)}.$$

Proof. Let $\epsilon > 0$, then there is a positive number δ such that $0 < \beta < \delta$ implies

$$\frac{\beta}{\varphi(\beta)} < \limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} + \epsilon.$$

In view of the way in which φ^* is defined, there is a family H of closed intervals such that:

$$A \subset \sigma(H),$$

$$\text{norm } H < \delta,$$

$$\sum_{J \in H} \varphi(|J|) < \varphi^*(A) + \epsilon.$$

We have, therefore,

$$\begin{aligned} |A| &\leq |\sigma(H)| \leq \sum_{J \in H} |J| = \sum_{J \in H} \left[\frac{|J|}{\varphi(|J|)} \varphi(|J|) \right] \\ &< \left(\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} + \epsilon \right) \sum_{J \in H} \varphi(|J|) \\ &< \left(\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} + \epsilon \right) (\varphi^*(A) + \epsilon). \end{aligned}$$

Our lemma clearly follows.

3.15. LEMMA. If I is a closed interval and A is a set with $\varphi^*(I \cdot A) < \infty$ and

$$\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} < \infty,$$

then

$$|I| \geq \varphi^*(I \cdot A) \limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)}.$$

Proof. If the right member of the relation just above is zero, the validity of the inequality is evident. Otherwise we may assume that each factor of the right member is positive. We let $0 < \epsilon < 1$, and fix δ so that $0 < \delta < \epsilon$, $0 < \varphi(\delta) < \infty$ and the two following relations hold:

$$\varphi^*(I \cdot A) > \varphi^*(I \cdot A)(1 - \epsilon) > 0,$$

$$\frac{\delta}{\varphi(\delta)} > \left[\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} \right] (1 - \epsilon) > 0.$$

Clearly, there is a family H of closed intervals, each of length δ , with

$$|I| + \epsilon > |I| + \delta > \sum_{J \in H} |J|,$$

and $I \cdot A \subset I \subset \sigma(H)$.

We have then the following inequality:

$$\begin{aligned}
 |I| + \epsilon &\geq \sum_{J \in H} |J| \\
 &= \sum_{J \in H} \frac{|J|}{\varphi(|J|)} \varphi(|J|) \\
 &\geq \left[\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} \right] (1 - \epsilon) \sum_{J \in H} \varphi(|J|) \\
 &\geq \left[\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} \right] (1 - \epsilon) \varphi^*(I \cdot A) \\
 &\geq \left[\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} \right] \varphi^*(I \cdot A) (1 - \epsilon)^2.
 \end{aligned}$$

The result is now evident.

3.16. LEMMA. If $\varphi^*(A) < \infty$ and

$$\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} < \infty,$$

then

$$|A| \geq \varphi^*(A) \limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)}.$$

Proof. Let $\epsilon > 0$. There is a family H of intervals such that $A \subset \sigma(H)$ and $|A| + \epsilon > \sum_{J \in H} |J|$. Since $\varphi^*(A) < \infty$, we have $\varphi^*(J \cdot A) < \infty$ for each $J \in H$, and recalling 3.15, we obtain the relation

$$\begin{aligned}
 |A| + \epsilon &> \sum_{J \in H} |J| \geq \sum_{J \in H} \varphi^*(J \cdot A) \limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} \\
 &\geq \varphi^*(A) \limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)}.
 \end{aligned}$$

The remainder of the proof is evident.

In view of 3.14 and 3.16 we can assert,

3.17. THEOREM. If $\varphi^*(A) < \infty$ and

$$\limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)} < \infty,$$

then

$$|A| = \varphi^*(A) \limsup_{\alpha \rightarrow 0} \frac{\alpha}{\varphi(\alpha)}.$$

4. The class $\mathcal{C}(\varphi, I)$.

4.1. DEFINITION. If H is a family of closed intervals and $\mathfrak{J}$ is a closed interval, then we define $H^*\mathfrak{J}$ as the subfamily of H such that $J \in H^*\mathfrak{J}$ if and only if $J \in H$ and $J \subset \mathfrak{J}$.

4.2. DEFINITION. We define the family $\mathcal{C}(\varphi, I)$ as follows: $H \in \mathcal{C}(\varphi, I)$ if and only if $\varphi \in P$, I is a closed interval, and H is a countable family of non-overlapping closed intervals with $\sigma(H) \subset I$ such that

$$\sum_{J \in H} \varphi(|J|) = \varphi(|I|)$$

and

$$\sum_{J \in H^* \mathfrak{J}} \varphi(|J|) \leq \varphi(|\mathfrak{J}|)$$

for each closed interval $\mathfrak{J}$.

4.3. DEFINITION. If H is a family of closed intervals then H^Δ is the family consisting of all closed intervals of the form $[a, b]$, where a is the left endpoint of some interval of H and b is the right endpoint of some interval of H .

4.4. DEFINITION. If H is a family of closed intervals then

$$\text{sh } H = \sum_{J \in H} (\text{interior of } J).$$

4.5. THEOREM. If $\varphi \in P$ and H is a countable family of closed intervals, then the conditions

$$(1) \quad \sum_{J \in H^* \mathfrak{J}} \varphi(|J|) \leq \varphi(|\mathfrak{J}|) \text{ for each closed interval } \mathfrak{J}, \text{ and}$$

$$(2) \quad \sum_{J \in H^* \mathfrak{J}'} \varphi(|J|) \leq \varphi(|\mathfrak{J}'|) \text{ for each } \mathfrak{J}' \in H^\Delta,$$

are equivalent.

Proof. Evidently (1) implies (2). Assuming (2), we establish the validity of (1) by a two-part argument.

Part I. If H' is a finite subfamily of H , and $\mathfrak{J}$ is a closed interval, then

$$\varphi(|\mathfrak{J}|) \geq \sum_{J \in H'^* \mathfrak{J}} \varphi(|J|).$$

Proof. If $H'^* \mathfrak{J}$ is vacuous, then

$$\sum_{J \in H'^* \mathfrak{J}} \varphi(|J|) = 0 \leq \varphi(|\mathfrak{J}|).$$

Otherwise, let $\mathfrak{J}' = [\inf \sigma(H'^* \mathfrak{J}), \sup \sigma(H'^* \mathfrak{J})]$. We have then $\mathfrak{J}' \subset \mathfrak{J}$ and since H' is a finite family, $\mathfrak{J}' \in (H'^* \mathfrak{J})^\Delta \subset H^\Delta$. Remembering that φ is non-decreasing and invoking (2), we may infer, with the help of the relation $H'^* \mathfrak{J}' = H'^* \mathfrak{J}$, that

$$\varphi(|\mathfrak{J}|) \geq \varphi(|\mathfrak{J}'|) \geq \sum_{J \in H'^* \mathfrak{J}'} \varphi(|J|) \geq \sum_{J \in H'^* \mathfrak{J}} \varphi(|J|) = \sum_{J \in H'^* \mathfrak{J}} \varphi(|J|).$$

This completes the proof of Part I.

Part II. If $\mathfrak{J}$ is a closed interval,

$$\varphi(|\mathfrak{J}|) \geq \sum_{J \in H^* \mathfrak{J}} \varphi(|J|).$$

Proof. By Part I

$$\varphi(|\mathfrak{J}|) \geq \sum_{J \in H'^* \mathfrak{J}} \varphi(|J|)$$

for each finite subfamily $H' \subset H$. We, therefore, conclude that the desired relation must hold and our proof is finished.

4.6. LEMMA. *If H_1 and H_2 are countable families of closed intervals such that $J' \in H_2$ implies $J' \subset J$ for some $J \in H_1$, and if $a_{J'} \geq 0$ for $J' \in H_2$, then*

$$\sum_{J' \in H_2} a_{J'} \leq \sum_{J \in H_1} \sum_{J' \in H_2^* J} a_{J'}.$$

Proof. Let $\delta(J', J) = 1$ if $J' \subset J$; let $\delta(J', J) = 0$ otherwise. We have then

$$\begin{aligned} (1) \quad \sum_{J \in H_1} \sum_{J' \in H_2^* J} a_{J'} &= \sum_{J \in H_1} \sum_{J' \in H_2} \delta(J', J) a_{J'} \\ &= \sum_{J' \in H_2} \sum_{J \in H_1} \delta(J', J) a_{J'} \\ &= \sum_{J' \in H_2} (a_{J'} \sum_{J \in H_1} \delta(J', J)). \end{aligned}$$

In view of our hypotheses, $J' \in H_2$ implies

$$(2) \quad \sum_{J \in H_1} \delta(J', J) \geq 1.$$

The proof is easily completed by combining (1) and (2).

4.7. LEMMA. *If H_1 is a countable non-overlapping family of closed intervals, H_2 is a countable family of closed intervals with $\text{sh } H_2 \subset \text{sh } H_1$, and if $a_{J'} \geq 0$ for $J' \in H_2$, then*

$$\sum_{J' \in H_2} a_{J'} = \sum_{J \in H_1} \sum_{J' \in H_2^* J} a_{J'}.$$

Proof. Using the notation of the preceding proof, we again have the relation

$$\sum_{J \in H_1} \sum_{J' \in H_2^* J} a_{J'} = \sum_{J' \in H_2} (a_{J'} \sum_{J \in H_1} \delta(J', J)).$$

Since $\text{sh } H_2 \subset \text{sh } H_1$ and H_1 is a non-overlapping family, $J' \in H_2$ implies $J' \subset J \in H_1$ for exactly one $J \in H_1$. It follows that

$$\sum_{J \in H_1} \delta(J', J) = 1,$$

and our lemma is evident.

4.8. THEOREM. *If $H_1 \in \mathcal{C}(\varphi, I)$, $H_2 \in \mathcal{C}(\varphi, I)$ and $\text{sh } H_2 \subset \text{sh } H_1$, then $H_2^* J \in \mathcal{C}(\varphi, J)$ for each $J \in H_1$.*

Proof. Since $H_2^* J \subset H_2$ for each $J \in H_1$ and since $H_2 \in \mathcal{C}(\varphi, I)$, we have

$$(1) \quad \sum_{J' \in (H_2^* J)^* \mathfrak{J}} \varphi(|J'|) \leq \sum_{J' \in H_2^* \mathfrak{J}} \varphi(|J'|) \leq \varphi(|\mathfrak{J}|),$$

for each closed interval $\mathfrak{J}$ and each $J \in H_1$; in particular, for each $J \in H_1$,

$$(2) \quad \sum_{J' \in H_2 * J} \varphi(|J'|) \leq \varphi(|J|).$$

In view of 4.7, we have

$$(3) \quad \sum_{J \in H_1} \sum_{J' \in H_2 * J} \varphi(|J'|) = \sum_{J' \in H_2} \varphi(|J'|).$$

Using (3), (2), and the fact that $H_1 \in \mathcal{C}(\varphi, I)$ and $H_2 \in \mathcal{C}(\varphi, I)$, we have

$$(4) \quad \begin{aligned} \varphi(|I|) &= \sum_{J \in H_2} \varphi(|J'|) \\ &= \sum_{J \in H_1} \sum_{J' \in H_2 * J} \varphi(|J'|) \leq \sum_{J \in H_1} \varphi(|J|) = \varphi(|I|). \end{aligned}$$

In view of (2) and (4), we conclude that

$$\sum_{J' \in (H_2 * J)} \varphi(|J'|) = \varphi(|J|),$$

for each $J \in H_1$. Recalling (1), we see that the proof is now complete.

4.9. LEMMA. *If $\epsilon > 0$, n is a positive integer, $H_n \in \mathcal{C}(\varphi, I)$, $H_{n+1} \in \mathcal{C}(\varphi, I)$, $\text{sh } H_{n+1} \subset \text{sh } H_n$ and V_n is a subfamily of H_n such that*

$$\sum_{J \in V_n} \varphi(|J|) > \varphi(|I|) - \epsilon,$$

then there is a finite subfamily V_{n+1} of H_{n+1} such that $\sigma(V_{n+1}) \subset \sigma(V_n)$ and

$$\sum_{J \in V_{n+1}} \varphi(|J|) > \varphi(|I|) - \epsilon.$$

Proof. Let

$$(1) \quad V = H_{n+1} \cdot E_{J'} \quad [J' \subset J \in V_n, \text{ for some } J].$$

Clearly, $J' \in V$ implies $J' \subset J \in V_n$ for some J and hence $\text{sh } V \subset \text{sh } V_n$ and

$$(2) \quad \sigma(V) \subset \sigma(V_n).$$

Now in view of 4.7

$$(3) \quad \sum_{J' \in V} \varphi(|J'|) = \sum_{J \in V_n} \sum_{J' \in V * J} \varphi(|J'|),$$

since V_n is a non-overlapping family. In the light of (1) we see that the relations

$$J' \in V * J, \quad J' \in H_{n+1} * J$$

are equivalent for each $J \in V_n$. We have then for each $J \in V_n$

$$(4) \quad \sum_{J' \in V * J} \varphi(|J'|) = \sum_{J' \in H_{n+1} * J} \varphi(|J'|).$$

Combining (3) and (4) and using 4.8, we have

$$(5) \quad \sum_{J' \in V} \varphi(|J'|) = \sum_{J \in V_n} \sum_{J' \in H_{n+1} \cdot J} \varphi(|J'|) = \sum_{J \in V_n} \varphi(|J|) > \varphi(|I|) - \epsilon.$$

Now if V_{n+1} is a sufficiently large finite subfamily of V , the theorem follows from (2) and (5).

4.10. THEOREM. *If*

(i) $H_1 = \{I\}$, $H_n \in \mathcal{C}(\varphi, I)$, $\text{sh } H_{n+1} \subset \text{sh } H_n$ ($n = 1, 2, 3, \dots$),

(ii) $\text{norm } H_n \rightarrow 0$ as $n \rightarrow \infty$,

(iii) $A = \sigma(H_1) \cdot \sigma(H_2) \cdot \sigma(H_3) \cdot \dots$,

then

$$\varphi^*(\mathfrak{J} \cdot A) \leq \varphi(|\mathfrak{J}|),$$

for each closed interval $\mathfrak{J}$ and

$$\varphi^*(A) = \varphi(|I|).$$

Proof. We divide the proof into parts.

Part I. $\varphi^*(\mathfrak{J} \cdot A) \leq \varphi(|\mathfrak{J}|)$ for each closed interval $\mathfrak{J}$.

Proof. Let $\mathfrak{J}$ be a fixed closed interval; let $\epsilon > 0$. Since $\varphi \in P$, there is a positive number β such that $0 < \alpha < \beta$ implies $\varphi(\alpha) < \epsilon$. Moreover, there is an integer N such that $\text{norm } H_N < \alpha$ and there are intervals J_1 and J_2 in H_N such that

$$\mathfrak{J} \cdot A \subset J_1 + J_2 + \sum_{J \in H_N \cdot \mathfrak{J}} J.$$

Since $H_N \in \mathcal{C}(\varphi, I)$, we conclude that

$$\begin{aligned} \varphi_\alpha^*(\mathfrak{J} \cdot A) &\leq \varphi(|J_1|) + \varphi(|J_2|) + \sum_{J \in H_N \cdot \mathfrak{J}} \varphi(|J|) \\ &\leq \varphi(|J_1|) + \varphi(|J_2|) + \varphi(|\mathfrak{J}|) \\ &\leq 2\varphi(\alpha) + \varphi(|\mathfrak{J}|) \\ &\leq 2\epsilon + \varphi(|\mathfrak{J}|). \end{aligned}$$

It is clear that this relation can be established for each positive $\alpha < \beta$, hence we have

$$\varphi^*(\mathfrak{J} \cdot A) \leq \varphi(|\mathfrak{J}|) + 2\epsilon.$$

It is also clear that this last relation must hold for each positive ϵ , and the proof of Part I is complete.

Part II. If $\epsilon > 0$, and G is a family of closed intervals with $A \subset \text{sh } G$, then for some positive integer N there is a subfamily $V \subset H_N$ such that $J' \in V$ implies $J' \subset (\text{interior of } J)$ for some $J \in G$ and

$$\varphi(|I|) \leq \epsilon + \sum_{J' \in V} \varphi(|J'|).$$

Proof. Let $V_1 = \{I\}$. Clearly V_1 is a subfamily of H_1 with

$$\sum_{J' \in V_1} \varphi(|J'|) = \varphi(|I|) > \varphi(|I|) - \epsilon.$$

Accordingly, 4.9 guarantees existence of finite families V_n ($n = 1, 2, 3, \dots$) such that $V_1 = H_1$, $V_n \subset H_n$, $\sigma(V_{n+1}) \subset \sigma(V_n)$, and

$$(1) \quad \sum_{J' \in V_n} \varphi(|J'|) > \varphi(|I|) - \epsilon$$

for $n = 1, 2, 3, \dots$. Using compactness, we can easily verify that for n sufficiently large, $J' \in V_n$ implies $J' \subset (\text{interior of } J)$ for some $J \in G$. This completes the proof of Part II.

Part III. $\varphi(|I|) \leq \varphi^*(A)$.

Proof. Let $\epsilon > 0$. In view of 3.10, there is a family G of closed intervals such that $A \subset \text{sh } G$ and

$$(1) \quad \sum_{J \in G} \varphi(|J|) \leq \varphi^*(A) + \epsilon.$$

By virtue of Part II and 4.6, there is a positive integer N and a subfamily $V \subset H_N$ such that

$$(2) \quad \varphi(|I|) \leq \epsilon + \sum_{J' \in V} \varphi(|J'|),$$

and

$$(3) \quad \sum_{J' \in V} \varphi(|J'|) \leq \sum_{J \in G} \sum_{J' \in V^*J} \varphi(|J'|).$$

Since $V \subset H_N$ and $H_N \in \mathcal{C}(\varphi, I)$, we have (recalling 4.8)

$$(4) \quad \sum_{J' \in V^*J} \varphi(|J'|) \leq \sum_{J' \in H_N^*J} \varphi(|J'|) \leq \varphi(|J|),$$

for each $J \in G$. Combining (2), (3), (4), and (1), we obtain the relation

$$\begin{aligned} \varphi(|I|) &\leq \epsilon + \sum_{J' \in V} \varphi(|J'|) \leq \epsilon + \sum_{J \in G} \sum_{J' \in V^*J} \varphi(|J'|) \\ &\leq \epsilon + \sum_{J \in G} \varphi(|J|) \leq \varphi^*(A) + 2\epsilon. \end{aligned}$$

In view of the arbitrariness of ϵ , the proof of Part III is complete.

Part IV. Summary. By Part I, $\varphi^*(\mathfrak{J} \cdot A) \leq \varphi(|\mathfrak{J}|)$ for each closed interval $\mathfrak{J}$. In particular, $\varphi^*(A) \leq \varphi(|I|)$ and in conjunction with Part III, this implies $\varphi^*(A) = \varphi(|I|)$. This completes the proof.

4.11. THEOREM. *If*

- (i) $H_n \in \mathcal{C}(\varphi, I)$, $H_1 = \{I\}$, $\text{sh } H_{n+1} \subset \text{sh } H_n$ ($n = 1, 2, 3, \dots$).
- (ii) $\text{norm } H_n \rightarrow 0$ as $n \rightarrow \infty$,

(iii) $A = \sigma(H_1) \cdot \sigma(H_2) \cdot \sigma(H_3) \cdot \dots$,

(iv) $\mathfrak{J}$ is a closed interval and m is a positive integer such that

$$\varphi(|\mathfrak{J}|) = \sum_{J \in H_m \star \mathfrak{J}} \varphi(|J|),$$

then

$$\varphi^*(\mathfrak{J} \cdot A) = \varphi(|\mathfrak{J}|).$$

Proof. By 4.10, $\varphi^*(\mathfrak{J} \cdot A) \leq \varphi(|\mathfrak{J}|)$. Also by 4.8 and 4.10

$$\varphi^*(J \cdot A) = \varphi(|J|)$$

for each $J \in H_m$. We have then

$$\varphi(|\mathfrak{J}|) = \sum_{J \in H_m \star \mathfrak{J}} \varphi(|J|) = \sum_{J \in H_m \star \mathfrak{J}} \varphi^*(J \cdot A) = \varphi^*\left(\sum_{J \in H_m \star \mathfrak{J}} J \cdot A\right) \leq \varphi^*(\mathfrak{J} \cdot A).$$

It follows that $\varphi^*(\mathfrak{J} \cdot A) = \varphi(|\mathfrak{J}|)$.

5. The role of convexity. We say a function φ is convex upward on an interval (a, b) if $(1 - \lambda)\varphi(a) + \lambda\varphi(b) \leq \varphi[(1 - \lambda)a + \lambda b]$ for $0 \leq \lambda \leq 1$, $0 \leq a < b < \infty$.

5.1. DEFINITION. $\varphi \in C$ if and only if φ is non-negative, continuous at the origin, and convex upward on its domain with $\varphi(1) > 0$ and $\varphi(0) = 0$.

5.2. THEOREM. If $\varphi \in C$, then it has the following properties:

5.2.1. If $0 \leq k \leq 1$ and $0 \leq t$, then

$$k\varphi(t) \leq \varphi(kt).$$

5.2.2. φ is non-decreasing.

5.2.3. If $t > 0$, then $\varphi(t) > 0$.

5.2.4. If a and b are non-negative numbers, then

$$\varphi(a + b) \leq \varphi(a) + \varphi(b).$$

5.2.5. If H is a finite family and $0 \leq a_x < \infty$ for $x \in H$, then

$$\varphi\left(\sum_{x \in H} a_x\right) \leq \sum_{x \in H} \varphi(a_x).$$

5.2.6. If a and b are positive numbers and $0 \leq c < a + b$, then

$$\varphi(c) < \varphi(a) + \varphi(b).$$

5.2.7. If $0 \leq t_1 \leq t_2$ and $0 \leq h$, then

$$\varphi(t_2 + h) - \varphi(t_2) \leq \varphi(t_1 + h) - \varphi(t_1).$$

5.2.8. φ is uniformly continuous.

Proof of 5.2.1. Since φ is convex upward and $\varphi(0) = 0$,

$$k\varphi(t) = (1 - k)\varphi(0) + k\varphi(t) \leq \varphi((1 - k)0 + kt) = \varphi(kt).$$

Proof of 5.2.2. Let $0 \leq t_1 \leq t_2 < t < \infty$. Since φ is convex upward,

$$\begin{aligned}\varphi(t_2) &= \varphi\left[\left(\frac{t-t_2}{t-t_1}\right)t_1 + \left(\frac{t_2-t_1}{t-t_1}\right)t\right] \\ &\geq \frac{t-t_2}{t-t_1}\varphi(t_1) + \frac{t_2-t_1}{t-t_1}\varphi(t) \\ &\geq \frac{t-t_2}{t-t_1}\varphi(t_1).\end{aligned}$$

It follows that

$$\varphi(t_2) \geq \lim_{t \rightarrow \infty} \frac{t-t_2}{t-t_1}\varphi(t_1) = \varphi(t_1).$$

Proof of 5.2.3. If $t \geq 1$, 5.2.2 implies

$$\varphi(t) \geq \varphi(1) > 0;$$

if $0 < t \leq 1$, 5.2.1 implies

$$0 < t\varphi(1) \leq \varphi(t).$$

Proof of 5.2.4. If $a + b = 0$, this is obvious. If $a + b > 0$, 5.2.1 implies

$$\begin{aligned}\frac{a}{a+b}\varphi(a+b) &\leq \varphi(a), \\ \frac{b}{a+b}\varphi(a+b) &\leq \varphi(b).\end{aligned}$$

The desired result is obtained by adding these two inequalities.

Proof of 5.2.5. This can be proved by induction using 5.2.4.

Proof of 5.2.6. If $c \leq a$ or $c \leq b$, the statement is an immediate consequence of 5.2.2 and 5.2.3; if $a < c$ and $b < c$, we have (using 5.2.3 and 5.2.1)

$$0 < \varphi(c) < \frac{a+b}{c}\varphi(c) = \frac{a}{c}\varphi(c) + \frac{b}{c}\varphi(c) \leq \varphi(a) + \varphi(b).$$

Proof of 5.2.7. Since φ is convex upward,

$$\begin{aligned}\frac{t_2-t_1}{t_2-t_1+h}\varphi(t_2+h) + \frac{h}{t_2-t_1+h}\varphi(t_1) \\ \leq \varphi\left[\frac{t_2-t_1}{t_2-t_1+h}\cdot(t_2+h) + \frac{h}{t_2-t_1+h}\cdot t_1\right] = \varphi(t_2),\end{aligned}$$

and

$$\begin{aligned}\frac{h}{t_2-t_1+h}\varphi(t_2+h) + \frac{t_2-t_1}{t_2-t_1+h}\varphi(t_1) \\ \leq \varphi\left[\frac{h}{t_2-t_1+h}\cdot(t_2+h) + \frac{t_2-t_1}{t_2-t_1+h}\cdot t_1\right] = \varphi(t_1+h).\end{aligned}$$

Adding these inequalities we have

$$\varphi(t_2 + h) + \varphi(t_1) \leq \varphi(t_2) + \varphi(t_1 + h).$$

The desired result is an immediate consequence.

Proof of 5.2.8. This follows directly from 5.2.2, 5.2.7, and the fact that φ is continuous at the origin.

5.3. THEOREM. *If $\varphi \in C$, I is a closed interval, H is a countable family of closed intervals with $\sigma(H) \subset I$, and*

$$\sum_{J \in H} \varphi(|J|) = \varphi(|I|),$$

then a necessary and sufficient condition that $H \in \mathcal{C}(\varphi, I)$ is that

$$\sum_{J \in H \cap \mathfrak{J}} \varphi(|J|) \leq \varphi(|\mathfrak{J}|), \text{ for each } \mathfrak{J} \in H^\Delta.$$

Proof. The condition is evidently necessary. We shall prove that it is also sufficient. Reviewing the definition of $\mathcal{C}(\varphi, I)$, we see that we need only show that H is a non-overlapping family and that

$$\sum_{J \in H \cap \mathfrak{J}} \varphi(|J|) \leq \varphi(|\mathfrak{J}|)$$

for each closed interval $\mathfrak{J}$. The latter statement follows at once from 4.5. Suppose, if possible, that $J_1 = [a_1, b_1]$ and $J_2 = [a_2, b_2]$ are closed intervals of the family H which overlap. More precisely, suppose $a_1 \leq a_2$ and $b_1 > a_2$. Let $\beta = \sup [b_1, b_2]$. We have then $[a_1, \beta] \in H^\Delta$ with $J_1 \subset [a_1, \beta]$ and $J_2 \not\subset [a_1, \beta]$. Since

$$(b_1 - a_1) + (b_2 - a_2) > (b_1 - a_1)$$

and

$$(b_1 - a_1) + (b_2 - a_2) = (b_2 - a_1) + (b_1 - a_2) > (b_2 - a_1),$$

we have

$$(b_1 - a_1) + (b_2 - a_2) > \beta - a_1.$$

Therefore, 5.2.6 implies

$$\varphi(\beta - a_1) < \varphi(b_1 - a_1) + \varphi(b_2 - a_2).$$

However, our condition implies (where $I^\Delta = [a_1, \beta]$)

$$\varphi(b_1 - a_1) + \varphi(b_2 - a_2) \leq \sum_{J \in H \cap I^\Delta} \varphi(|J|) \leq \varphi(\beta - a_1).$$

These last two relations are clearly incompatible, and we, therefore, conclude that H is a non-overlapping family. This completes the proof.

5.4. THEOREM. If $\varphi \in C$, I is a closed interval, H is a countable family of closed intervals with $\sigma(H) \subset I$ and

$$\sum_{J \in H} \varphi(|J|) = \varphi(|I|),$$

then a necessary and sufficient condition that $H \in \mathcal{C}(\varphi, I)$ is that

$$\sum_{J \in H} \varphi(|J \cdot \mathfrak{J}|) \leq \varphi(|\mathfrak{J}|),$$

for each closed interval $\mathfrak{J}$.

Proof. The sufficiency of this condition is an immediate consequence of 5.3; we proceed with a proof of its necessity.

Fix $\mathfrak{J} = [a, b]$ with $a < b$. We introduce the following notation: if $x \in J \in H$, x' and x'' are respectively the left and right endpoints of J ; otherwise, let $x' = x'' = x$. If $a' = b'$ and $a'' = b''$, then $\mathfrak{J} \subset [a', b''] \in H$ and we have,

$$(1) \quad \sum_{J \in H} \varphi(|J \cdot \mathfrak{J}|) = \varphi(|\mathfrak{J}|).$$

Alternatively, we must have the henceforth assumed relation

$$(2) \quad a' \leq a \leq a'' \leq b' \leq b \leq b'',$$

and we complete the proof by establishing relation (7) below. We let $[a', b''] = I'$ and $[a'', b'] = I''$, then in view of the validity of the two relations

$$\sum_{J \in H \cap I'} \varphi(|J \cdot \mathfrak{J}|) = \sum_{J \in H \cap I''} \varphi(|J|) + \varphi(a'' - a) + \varphi(b - b'),$$

and

$$\sum_{J \in H \cap I'} \varphi(|J|) = \sum_{J \in H \cap I''} \varphi(|J|) + \varphi(a'' - a') + \varphi(b'' - b'),$$

which are evident from a figure, we have

$$\begin{aligned} (3) \quad \sum_{J \in H} \varphi(|J \cdot \mathfrak{J}|) &= \sum_{J \in H \cap I'} \varphi(|J \cdot \mathfrak{J}|) \\ &= \sum_{J \in H \cap I''} \varphi(|J|) - \varphi(a'' - a') \\ &\quad + \varphi(a'' - a) - \varphi(b'' - b') + \varphi(b - b'). \end{aligned}$$

Since $H \in \mathcal{C}(\varphi, I)$, we have

$$(4) \quad \sum_{J \in H \cap I'} \varphi(|J|) \leq \varphi(b'' - a').$$

In view of (2), 5.2.7 implies

$$(5) \quad \varphi(a'' - a') - \varphi(a'' - a) \geq \varphi(b'' - a') - \varphi(b'' - a),$$

and

$$(6) \quad \varphi(b'' - b') - \varphi(b - b') \geq \varphi(b'' - a) - \varphi(b - a).$$

Combining relations (3), (4), (5), and (6), we have

$$\begin{aligned}
 (7) \quad \sum_{J \in H} \varphi(|J \cdot \mathfrak{J}|) &\leq \varphi(b'' - a') - \varphi(b'' - a') \\
 &\quad + \varphi(b'' - a) - \varphi(b'' - a) + \varphi(b - a) \\
 &= \varphi(b - a) = \varphi(|\mathfrak{J}|).
 \end{aligned}$$

5.5. THEOREM. *If $\varphi \in C$, $H_1 \in \mathcal{C}(\varphi, I)$ and H_2 is a countable family of closed intervals with $\text{sh } H_2 \subset \text{sh } H_1$, then a necessary and sufficient condition that $H_2 \in \mathcal{C}(\varphi, I)$ is that*

$$(1) \quad H_2 * J \in \mathcal{C}(\varphi, J) \text{ for each } J \in H_1.$$

Proof. Necessity of the condition is a direct consequence of 4.8. We shall show that the condition is sufficient. In view of our hypotheses and (1), H_2 is a countable non-overlapping family of closed intervals with $\sigma(H_2) \subset I$. Using 4.7, (1), and the fact that $H_1 \in \mathcal{C}(\varphi, I)$, we obtain the relation

$$\sum_{J' \in H_2} \varphi(|J'|) = \sum_{J \in H_1} \sum_{J' \in H_2 * J} \varphi(|J'|) = \sum_{J \in H_1} \varphi(|J|) = \varphi(|I|).$$

Recalling 4.5, we see that we may complete the proof by establishing the following statement:

$$(2) \quad \text{If } \mathfrak{J} \in H_2^\Delta, \text{ then } \sum_{J' \in H_2 * \mathfrak{J}} \varphi(|J'|) \leq \varphi(|\mathfrak{J}|).$$

We let $[a, b] = \mathfrak{J} \in H_2^\Delta$. If $\mathfrak{J} \subset J \in H_1$, then $\mathfrak{J} \in (H_2 * J)^\Delta$, and (1) implies (2). Otherwise, in the notation of the proof of 5.4, there are intervals $[a', a'']$ and $[b', b'']$ which are in H_1 with $a' \leq a < a'' \leq b' < b \leq b''$. Recalling that $\mathfrak{J} \in H_2^\Delta$ and $\text{sh } H_2 \subset \text{sh } H_1$ and using 4.7, the fact that $H_2 * J \in \mathcal{C}(\varphi, J)$ for each $J \in H_1$, and 5.4, we have

$$\begin{aligned}
 \sum_{J' \in H_2 * \mathfrak{J}} \varphi(|J'|) &= \sum_{J' \in H_2} \varphi(|J' \cdot \mathfrak{J}|) \\
 &= \sum_{J' \in H_2} [\varphi(|J' \cdot [a, a'']|) + \varphi(|J' \cdot [a'', b']|) \\
 &\quad + \varphi(|J' \cdot [b', b]|)] \\
 &= \sum_{J' \in H_2} \varphi(|J' \cdot [a, a'']|) + \sum_{J \in H_1} \sum_{J' \in H_2 * J} \varphi(|J' \cdot [a'', b']|) \\
 &\quad + \sum_{J' \in H_2} \varphi(|J' \cdot [b', b]|) \\
 &\leq \varphi(|[a, a'']|) + \sum_{J \in H_1} \varphi(|J \cdot [a'', b']|) + \varphi(|[b', b]|) \\
 &= \varphi(|[a', a''] \cdot \mathfrak{J}|) + \sum_{J \in H_1 * \mathfrak{J}} \varphi(|J|) + \varphi(|[b', b''] \cdot \mathfrak{J}|) \\
 &= \sum_{J \in H_1} \varphi(|J \cdot \mathfrak{J}|) \leq \varphi(|\mathfrak{J}|).
 \end{aligned}$$

This completes the proof.

5.6. THEOREM. If $\varphi \in C$, $I = [a, b]$ is a closed interval and H is a countable family of non-overlapping closed intervals with $\sigma(H) \subset I$ and

$$\sum_{J \in H} \varphi(|J|) = \varphi(|I|),$$

then a sufficient condition that $H \in \mathcal{C}(\varphi, I)$ is that

$$\sum_{J \in H^* \mathfrak{J}} \varphi(|J|) = \varphi(|\mathfrak{J}|)$$

for each interval $\mathfrak{J} \in H'$, where H' is the family of all intervals of the form $[a, x]$ where $[y, x] \in H^\Delta$ for some y .

Proof. In view of 4.5, we need only establish the following assertion:

$$(1) \quad \text{If } \mathfrak{J} \in H^\Delta, \text{ then } \sum_{J \in H^* \mathfrak{J}} \varphi(|J|) \leq \varphi(|\mathfrak{J}|).$$

To this end, let $\mathfrak{J} = [\alpha, \beta] \in H^\Delta$. Then there is an interval $[\alpha, \gamma] \in H$ such that the families $H^*[\alpha, \beta]$ and $H^*[a, \gamma]$ have only this interval in common. Moreover, since H is a non-overlapping family, we have

$$\begin{aligned} (2) \quad \sum_{J \in H^* \mathfrak{J}} \varphi(|J|) &= \sum_{J \in H^*[\alpha, \beta]} \varphi(|J|) \\ &= \sum_{J \in H^*[\alpha, \beta]} \varphi(|J|) - \sum_{J \in H^*[a, \gamma]} \varphi(|J|) + \varphi(\gamma - \alpha). \end{aligned}$$

We now observe that $[a, \beta]$ and $[a, \gamma]$ are in H' and hence that our condition implies the relations

$$(3) \quad \sum_{J \in H^*[\alpha, \beta]} \varphi(|J|) = \varphi(\beta - a),$$

and

$$(4) \quad \sum_{J \in H^*[a, \gamma]} \varphi(|J|) = \varphi(\gamma - a).$$

Combination of (2), (3), and (4) yields the following relation:

$$(5) \quad \sum_{J \in H^* \mathfrak{J}} \varphi(|J|) = \varphi(\beta - a) - \varphi(\gamma - a) + \varphi(\gamma - \alpha).$$

Recalling that $a \leq \alpha < \gamma \leq \beta$, we see that

$$0 < \gamma - \alpha \leq \beta - \alpha$$

and

$$(\beta - a) - (\beta - \alpha) = (\gamma - a) - (\gamma - \alpha) = \alpha - a \geq 0.$$

Therefore, convexity implies (see 5.2.7),

$$\varphi(\beta - a) - \varphi(\beta - \alpha) \leq \varphi(\gamma - a) - \varphi(\gamma - \alpha).$$

Combining this last inequality with (5), we have

$$\sum_{J \in H^*_{\mathfrak{J}}} \varphi(|J|) \leq \varphi(\beta - \alpha) = \varphi(|\mathfrak{J}|).$$

Our proof is complete.

5.7. THEOREM. *If $\varphi \in C$, $I = [a, b]$ is a closed interval and $B \subset I$ is a closed set of Lebesgue measure zero with a and $b \in B$, then there is a family $H \in \mathcal{C}(\varphi, I)$ such that for $c \in B$*

$$\sum_{\mathfrak{J} \in H^*[a, c]} \varphi(|\mathfrak{J}|) = \varphi(c - a),$$

and

$$H^*[a, c] \in \mathcal{C}(\varphi, [a, c]).$$

Proof. Let $\mathfrak{J}$ represent the disjointed family of all open sub-intervals of $[a, b]$ which are contiguous to B . Corresponding to each $J \in \mathfrak{J}$, we let J' be an interval $[\gamma, \beta]$ for which $(\alpha, \beta) = J$ and

$$\varphi(\beta - \gamma) = \varphi(\beta - a) - \varphi(\alpha - a),$$

where γ is the sup of the closed set

$$[\alpha, \beta] \cdot E_x[\varphi(\beta - x) = \varphi(\beta - a) - \varphi(\alpha - a)].$$

This set is nonvacuous in view of the relation

$$\varphi(\beta - \beta) \leq \varphi(\beta - a) - \varphi(\alpha - a) \leq \varphi(\beta - \alpha).$$

For each $J \in \mathfrak{J}$, it is clear that J' is contained in the closure of J , that the right endpoint of J' agrees with that of J , and that

$$\int_J \varphi'(t - a) dt = \varphi(|J'|).$$

Let

$$K = \sum_{J \in \mathfrak{J}} \{J'\}; \quad H = K \cdot E[\mathfrak{J} > 0].$$

Note that the right endpoint of each interval of H' (in the notation of 5.6) is in B and that H is a non-overlapping family. Then, since $|B| = 0$, we have for $c \in B$

$$\begin{aligned} \sum_{\mathfrak{J} \in H^*[a, c]} \varphi(|\mathfrak{J}|) &= \sum_{J' \in K^*[a, c]} \varphi(|J'|) = \sum_{J \in \mathfrak{J}^*[a, c]} \varphi(|J'|) \\ &= \sum_{J \in \mathfrak{J}^*[a, c]} \int_J \varphi'(t - a) dt \\ &= \int_a^c \varphi'(t - a) dt = \varphi(c - a). \end{aligned}$$

In view of 5.6, the proof is complete.

5.8. DEFINITION. $\mathcal{C}'(\varphi, I)$ is the family defined by $H \in \mathcal{C}'(\varphi, I)$ if and only if $\varphi \in C$, $I = [a, b]$ is a closed interval, H is a family of closed intervals such that

$$H = \sum_{m=1}^n \left\{ \left[a + \frac{m-1}{n-1} (\lambda - \mu), a + \mu + \frac{m-1}{n-1} (\lambda - \mu) \right] \right\},$$

where n is an integer not less than 2, $n\varphi(\mu) = \varphi(b - a)$ and $\lambda = b - a$.

5.9. THEOREM. $\mathcal{C}'(\varphi, I) \subset \mathcal{C}(\varphi, I)$.

Proof. Let $H \in \mathcal{C}'(\varphi, I)$. Clearly H is a non-overlapping family of closed intervals with $\sigma(H) \subset I$. Let $[\alpha, \beta] \in H^\Delta$. We have then

$$\beta = a + \mu + \frac{m_2 - 1}{n - 1} (\lambda - \mu),$$

$$\alpha = a + \frac{m_1 - 1}{n - 1} (\lambda - \mu),$$

where $n \geq m_2 > m_1 \geq 1$. We let $N = m_2 - m_1 + 1$. We have then

$$(1) \quad \sum_{J \in H^\Delta[\alpha, \beta]} \varphi(|J|) = N\varphi(\mu).$$

We have also, since φ is convex,

$$\begin{aligned} \varphi(\beta - \alpha) &= \varphi\left(\mu + \frac{m_2 - m_1}{n - 1} (\lambda - \mu)\right) \\ &= \varphi\left(\mu + \frac{N - 1}{n - 1} (\lambda - \mu)\right) \\ &= \varphi\left(\left(1 - \frac{N - 1}{n - 1}\right)\mu + \frac{N - 1}{n - 1} \lambda\right) \\ (2) \quad &\geq \left(1 - \frac{N - 1}{n - 1}\right)\varphi(\mu) + \left(\frac{N - 1}{n - 1}\right)\varphi(\lambda) \\ &= \left(1 - \frac{N - 1}{n - 1} + \left(\frac{N - 1}{n - 1}\right)n\right)\varphi(\mu) \\ &= N\varphi(\mu). \end{aligned}$$

Also clearly,

$$(3) \quad \sum_{J \in H^\Delta[a, b]} \varphi(|J|) = n\varphi(\mu) = \varphi(\lambda) = \varphi(|I|).$$

In view of (1), (2), and (3), the remainder of the proof is evident.

5.10. DEFINITION. If $\varphi \in C$, $I = [a, b]$ is a closed interval and n is an integer not less than 2, then $\mathfrak{R}(\varphi, I, n)$ is defined as that family H such that

$$H = \sum_{m=1}^n \left\{ \left[a + \frac{m-1}{n-1} (\lambda - \mu), a + \mu + \frac{m-1}{n-1} (\lambda - \mu) \right] \right\},$$

where $\lambda = b - a$ and $\mu = \inf E_t[\varphi(t) = \varphi(\lambda)/n]$.

By virtue of 5.8, the following theorem is now evident.

5.11. THEOREM. If $\varphi \in C$, I is a closed interval, and n is an integer which exceeds 1, then $\mathfrak{R}(\varphi, I, n) \in \mathcal{C}'(\varphi, I)$.

6. The φ -Cantorian functions and their convex moduli.

6.1. DEFINITION. We say that A is a φ -Cantorian set if and only if $\varphi \in C$ and there are families $H_1, H_2, H_3, \dots$, and a closed interval I such that:

6.1.1. $H_n \in \mathcal{C}(\varphi, I)$ and $\text{sh } H_{n+1} \subset \text{sh } H_n$, for $n \geq 1$;

6.1.2. $\text{norm } H_n \rightarrow 0$ as $n \rightarrow \infty$;

6.1.3. $A = \sigma(H_1) \cdot \sigma(H_2) \cdot \sigma(H_3) \cdot \dots$.

If A is φ -Cantorian, 4.10 implies $\varphi^*(A) < \infty$.

6.2. DEFINITION. We say that f is a φ -Cantorian function if and only if the domain of f is R and there is a φ -Cantorian set A for which the following relation holds:

$$f(x) = \varphi^*((-\infty, x] \cdot A) \quad (x \in R).$$

6.3. CONVENTION. A function f will be spoken of as the φ -Cantorian function associated with A if and only if f is a φ -Cantorian function and

$$f(x) = \varphi^*((-\infty, x] \cdot A) \quad (x \in R).$$

6.4. DEFINITION. $\varphi \in C'$ if and only if φ is a function on $[0, \infty)$ to $[0, \infty)$ and is convex upward with $\varphi(0) = 0$.

6.5. DEFINITION. Whenever f is a function on $E \subset R$ to R , we define the convex modulus of f as the functional Φ such that:

6.5.1. $\Phi \in C'$,

6.5.2. $\Phi(t) = \inf_{\varphi \in C''} \varphi(t)$, for $0 \leq t < \infty$,

where $C'' = C' \cdot E_\varphi[|f(t_2) - f(t_1)| \leq \varphi(t_2 - t_1) \text{ for } t_1 \in E, t_2 \in E]$.

If Φ is the convex modulus of f , then $\Phi(0) = 0$. A function f can readily be found such that the convex modulus of f does not exist. However, a necessary and sufficient condition that the convex modulus of f be continuous at the origin is that f be uniformly continuous.

By making suitable use of covering theorems, it is possible to establish 6.6 and 6.7 below. We do not give the proofs nor do we make any further use of them.

6.6. THEOREM. If $\varphi \in C$ and $\varphi^*(A) < \infty$, then for some set $B \subset A$ with $\varphi^*(A) = \varphi^*(B)$, we have for $x \in B$

$$\limsup_{|J| \rightarrow 0, J \in F(x)} \frac{\varphi^*(J \cdot A)}{\varphi(|J|)} = 1,$$

where $F(x)$ is the family of all those closed intervals of which x is an element.

6.7. THEOREM. If f is a φ -Cantorian function and Φ is the convex modulus of f , then

$$\limsup_{t \rightarrow 0} \frac{\Phi(t)}{\varphi(t)} \geq 1.$$

6.8. DEFINITION. If $\varphi \in C'$ and $B \subset R$, then $\langle \varphi, B \rangle$ is the least function in C' which dominates φ on B .

Clearly $\langle \varphi, B \rangle$ exists whenever $\varphi \in C'$ and $B \subset R$, and $\langle \varphi, B \rangle$ and φ must coincide on B . We also notice that the function $\langle \varphi, B \rangle$ and $\langle \langle \varphi, B \rangle, B \rangle$ are identical.

6.9. THEOREM. If A is a φ -Cantorian set, f is the φ -Cantorian function associated with A , Φ is the convex modulus of f , and B is any subset of the set of all numbers of the form $|J|$ where J is a closed interval such that $\varphi^*(A \cdot J) = \varphi(|J|)$, then

$$\langle \varphi, B \rangle(t) \leq \Phi(t) \leq \varphi(t) \quad (0 \leq t < \infty).$$

Proof. We divide the proof into parts.

Part I.
$$\Phi(t) \leq \varphi(t) \quad (0 \leq t < \infty).$$

Proof. Let $t_1 \leq t_2$; then recalling 6.3, 6.1, and 4.10, we have

$$\begin{aligned} |f(t_2) - f(t_1)| &= f(t_2) - f(t_1) \\ &= \varphi^*((-\infty, t_2] \cdot A) - \varphi^*((-\infty, t_1] \cdot A) \\ &= \varphi^*([t_1, t_2] \cdot A) \\ &\leq \varphi(|t_2 - t_1|). \end{aligned}$$

Therefore, in the notation of 6.5, $\varphi \in C''$ and it follows that

$$\Phi(t) \leq \varphi(t) \quad (0 \leq t < \infty).$$

Part II.
$$\langle \varphi, B \rangle(t) \leq \Phi(t) \quad (0 \leq t < \infty).$$

Proof. If $t \in B$, then there is a closed interval $[a, a + t]$ such that

$$\varphi(t) = \varphi^*([a, a + t] \cdot A).$$

This implies

$$\begin{aligned} \varphi(t) &= \varphi^*((-\infty, a + t] \cdot A) - \varphi^*((-\infty, a] \cdot A) \\ &= f(a + t) - f(a) \\ &\leq \Phi(t). \end{aligned}$$

That is to say, Φ dominates φ on B . Moreover, since Φ is the convex modulus of f , $\Phi \in C'$ and recalling 6.8, we see that

$$\langle \varphi, B \rangle(t) \leq \Phi(t) \quad (0 \leq t < \infty).$$

This completes the proof of Part II. Combining Parts I and II, we have our theorem.

6.10. THEOREM. *If A is a φ -Cantorian set, f is the φ -Cantorian function associated with A , Φ is the convex modulus of f , and if in the notation of 6.1*

- (i) $H_1 = \{I\}$,
- (ii) $H_{n+1}^* J \in \mathcal{C}'(\varphi, J)$, for each $J \in H_n$ for $n \geq 1$,
- (iii) $B = \sum_{J \in H} \{|J|\}$, where $H = H_1 + H_2 + H_3 + \dots$,

then

$$\langle \varphi, B \rangle(t) = \Phi(t) \quad (0 \leq t < \infty).$$

Proof. We divide the proof into parts.

Part I. $H_n \in \mathcal{C}(\langle \varphi, B \rangle, I)$ ($n \geq 1$).

Proof. Let $J = [a, b] \in H_n$; then by virtue of 5.8

$$H_{n+1}^* J = \sum_{m=1}^k \left\{ \left[a + \frac{m-1}{k-1} (\lambda - \mu), a + \mu + \frac{m-1}{k-1} (\lambda - \mu) \right] \right\},$$

where $k \geq 2$, $\lambda = b - a$, and $k\varphi(\mu) = \varphi(\lambda)$. Clearly $\mu \in B$ and $\lambda \in B$ and hence

$$k\langle \varphi, B \rangle(\mu) = \langle \varphi, B \rangle(\lambda),$$

since $\langle \varphi, B \rangle$ and φ must coincide on B . We conclude, therefore, that $H_{n+1}^* J \in \mathcal{C}'(\langle \varphi, B \rangle, J)$ and in view of 5.9, we also see that $H_{n+1}^* J \in \mathcal{C}(\langle \varphi, B \rangle, J)$. Now if $H_n \in \mathcal{C}(\langle \varphi, B \rangle, I)$, 5.5 and the preceding statement imply $H_{n+1} \in \mathcal{C}(\langle \varphi, B \rangle, I)$ and since it is evident that $H_1 \in \mathcal{C}(\langle \varphi, B \rangle, I)$, our assertion follows by induction. This proves Part I.

Part II. *If $F(x) = \langle \varphi, B \rangle^*((-\infty, x] \cdot A)$, for $x \in R$, then $F(x) = f(x)$ for $x \in R$.*

Proof. Let $E = A^\square + E_x$ [for some n and some $J \in H_n$, $J = [x, x+h]$, where $h > 0$]. If $x \in E$ and the set $(-\infty, x] \cdot A$ contains not more than one point, then $f(x) = F(x) = 0$. If $x \in E$ and $(-\infty, x] \cdot A$ contains more than one point, then for some n

$$H_n^*(-\infty, x] \neq \emptyset \text{ and } A \cdot (-\infty, x] \subset \sigma(H_n^*(-\infty, x]) + \{x\}.$$

We let $H_n^*(-\infty, x] = U$. In view of Part I, we see that 4.8 and 4.10 imply the relations:

$$(1) \quad f(x) = \varphi^*((-\infty, x] \cdot A) = \sum_{J \in U} \varphi^*(J \cdot A) = \sum_{J \in U} \varphi(|J|)$$

and

$$(2) \quad F(x) = \langle \varphi, B \rangle^*((-\infty, x] \cdot A) = \sum_{J \in U} \langle \varphi, B \rangle^*(J \cdot A) = \sum_{J \in U} \langle \varphi, B \rangle(|J|).$$

Since $|J| \in B$ for each $J \in H_n$, the right members of these two relations are equal. Since the set E is everywhere dense in R and the functions F and f are continuous, the proof of Part II is complete.

Part III. $\langle \varphi, B \rangle(t) = \Phi(t) \quad (0 \leq t < \infty).$

Proof. In view of Part II, Φ is the convex modulus of F as well as of f . Part I implies that the set A is a $\langle \varphi, B \rangle$ -Cantorian set and hence it follows from 6.9 that

$$\langle \langle \varphi, B \rangle, B \rangle(t) \leq \Phi(t) \leq \langle \varphi, B \rangle(t) \quad (0 \leq t < \infty).$$

Since the functions $\langle \langle \varphi, B \rangle, B \rangle$ and $\langle \varphi, B \rangle$ are clearly identical, the proof of Part III and of the theorem is complete.

6.11. CONSTRUCTION. Let $\varphi \in C$ and I be a closed interval. Let $H_1 = \{I\}$ and for $n \geq 2$ let (see 5.10)

$$H_n = \sum_{J \in H_{n-1}} \mathfrak{R}(\varphi, J, k(n)),$$

where $k(n) \geq 2$. Finally, let $A = \sigma(H_1) \cdot \sigma(H_2) \cdot \dots$.

Evidently, H_n is a family of non-overlapping closed intervals with $\sigma(H_n) \subset I$ and $\text{sh } H_{n+1} \subset \text{sh } H_n$ for each n . Furthermore, by 5.10, 5.8, and 5.9 $H_n^* J \in \mathcal{C}(\varphi, J)$ for each $J \in H_{n-1}$ and, consequently, since $H_1 \in \mathcal{C}(\varphi, I)$ we may use 5.5 to show that $H_n \in \mathcal{C}(\varphi, I)$ for each n by induction. Also

$$\text{norm } H_n = \inf_t E \left[\varphi(t) = \frac{\varphi(\text{norm } H_{n-1})}{k(n)} \right]$$

and since $\varphi \in C$, we have

$$\begin{aligned} \text{norm } H_n &\leq \frac{\varphi(\text{norm } H_n)}{\varphi(1)} = \frac{\varphi(\text{norm } H_{n-1})}{\varphi(1)k(n)} \\ &= \frac{\varphi(|I|)}{\varphi(1) \cdot k(2) \cdot k(3) \cdot \dots \cdot k(n)} \leq \frac{\varphi(|I|)}{\varphi(1)(2)^{n-1}}. \end{aligned}$$

It follows that $\text{norm } H_n \rightarrow 0$ as $n \rightarrow \infty$.

In view of these facts, A is a φ -Cantorian set. Moreover, by 4.10, $\varphi^*(A) = \varphi(|I|)$ and if f is the φ -Cantorian function associated with A , and Φ is its convex modulus, then 6.10 implies $\langle \varphi, B \rangle(t) = \Phi(t)$ for $0 \leq t < \infty$ where B is the set defined in 6.10.

6.12. CONSTRUCTION. Let $\varphi \in C$ and $E \subset R$ be a bounded, nonvacuous closed set of Lebesgue measure zero; let $I = [a, b]$ where $a = \inf E$ and $b = \sup E$. Let $H_1 = \{I\}$, H_2 be the family whose existence is guaranteed by 5.7, and for $n \geq 3$, let

$$H_n = \sum_{J \in H_{n-1}} \mathfrak{R}(\varphi, J, k(n)),$$

where $k(n) \geq 2$. Let $A = \sigma(H_1) \cdot \sigma(H_2) \cdot \dots$.

By reasoning almost identical to that given following 6.11, we may conclude that A is a φ -Cantorian set with $\varphi^*(A) = \varphi(|I|)$. If $c \in E$, then by 5.7, $H_2^*[a, c] \in \mathcal{C}(\varphi, [a, c])$ and recalling 5.5 and 4.10, we see that $\varphi^*([a, c] \cdot A) =$

$\varphi(c - a)$. Consequently, if f is the φ -Cantorian function associated with A , and Φ is its convex modulus, 6.9 implies that

$$\langle \varphi, E' \rangle(t) \leq \Phi(t) \leq \varphi(t) \quad (0 \leq t < \infty),$$

where E' is the set defined as follows:

$$E' = E_x[(x + a) \mathfrak{x} E].$$

6.13. THEOREM. *If A is a φ -Cantorian set with $|A| = 0$, and if f is the associated φ -Cantorian function, then f is singular.*

Proof. Since A is φ^* -measurable, we have $A = C_0 + C_1 + C_2 + \cdots$, where $\varphi^*(C_0) = 0$ and the sets C_n are closed for $n \geq 1$. For each n , let

$$f_n(x) = \varphi^*((-\infty, x] \cdot C_n) \quad (x \mathfrak{x} R).$$

Clearly, $0 \leq f_0(x) \leq \varphi^*(C_0) = 0$, and we have

$$f(x) = \varphi^*((-\infty, x] \cdot A) = \sum_{n=1}^{\infty} f_n(x).$$

Since C_n is a closed set with $|C_n| = 0$, f_n is singular for $n \geq 1$. It now follows that f is singular from a well-known result [10; 168].

7. A covering problem.

7.1. THEOREM. *If I is a closed interval and $L_n > 0$ for $n \geq 1$ with*

$$\sum_{n=1}^{\infty} L_n = L < |I|,$$

then there is a set $A \subset I$ with $|A| = 0$ which is not a subset of

$$\sum_{n=1}^{\infty} J_n,$$

whenever J_n is a closed interval with $|J_n| = L_n$ for each $n \geq 1$.

Proof. Let $\epsilon = |I| - L$. For each positive integer k there is a positive integer $N(k)$ such that

$$\sum_{n=N(k)}^{\infty} L_n < \frac{\epsilon}{2^{k+1}}.$$

For $k \geq 1$, let

$$\begin{aligned} \varphi_k(x) &= x & \left(0 \leq x \leq \frac{\epsilon}{2^{k+1}N(k)}\right), \\ \varphi_k(x) &= \frac{\epsilon}{2^{k+1}N(k)} & \left(\frac{\epsilon}{2^{k+1}N(k)} \leq x < \infty\right). \end{aligned}$$

Let $\varphi_0(x) = x$ for $0 \leq x < \infty$ and finally let

$$\varphi(x) = \sum_{k=0}^{\infty} \varphi_k(x) \quad (0 \leq x < \infty).$$

It is clear that φ_k is of type C with $\varphi'_k(0) = 1$ for each non-negative k and hence φ must be of type C with $\varphi'(0) = +\infty$.

Now in view of 6.11 and the remarks following, there is a set $A \subset I$ which is φ -Cantorian with

$$(1) \quad \varphi^*(A) = \varphi(|I|) = \varphi(L + \epsilon) > \varphi_0(L + \epsilon) = L + \epsilon.$$

Also, 3.14 implies $|A| = 0$. If

$$A \subset \sum_{n=1}^{\infty} J_n,$$

where $|J_n| = L_n$ for each $n \geq 1$, 4.10 and (1) imply

$$(2) \quad \sum_{n=1}^{\infty} \varphi(|J_n|) \geq \sum_{n=1}^{\infty} \varphi^*(A \cdot J_n) \geq \varphi^*(A) > L + \epsilon.$$

Recalling our definition of φ and φ_k , we see that we must also have

$$\begin{aligned} \sum_{n=1}^{\infty} \varphi(|J_n|) &= \sum_{n=1}^{\infty} \varphi(L_n) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \varphi_k(L_n) \right) \\ &= \sum_{n=1}^{\infty} (\varphi_0(L_n) + \sum_{k=1}^{\infty} \varphi_k(L_n)) \\ &= \sum_{n=1}^{\infty} L_n + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \varphi_k(L_n) \\ &= L + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \varphi_k(L_n) \\ (3) \quad &= L + \sum_{k=1}^{\infty} \left(\sum_{n=1}^{N(k)} \varphi_k(L_n) + \sum_{n=N(k)+1}^{\infty} \varphi_k(L_n) \right) \\ &\leq L + \sum_{k=1}^{\infty} \left(N(k) \frac{\epsilon}{2^{k+1}N(k)} + \sum_{n=N(k)+1}^{\infty} L_n \right) \\ &= L + \sum_{k=1}^{\infty} \left(\frac{\epsilon}{2^{k+1}} + \sum_{n=N(k)+1}^{\infty} L_n \right) \\ &< L + \sum_{k=1}^{\infty} \left(\frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^{k+1}} \right) = L + \epsilon. \end{aligned}$$

Relations (2) and (3) are clearly incompatible and our theorem follows.

8. A construction theorem.

8.1. THEOREM. If $0 < \alpha \leq 1$, $\Lambda > 0$, $\varphi(x) = \Lambda x^\alpha$ for $0 \leq x < \infty$, and I is a closed interval, then there is a φ -Cantorian set $A \subset I$ which is of Hausdorff dimension α , and if f is the φ -Cantorian function associated with A , f satisfies a Lipschitz condition of order α and modulus Λ .

Proof. Clearly $\varphi \in C$ and in view of 6.11, there is a φ -Cantorian set $A \subset I$ with $\varphi^*(A) = \varphi(|I|)$. This set is evidently of dimension α . Theorem 4.10 implies the remainder of our conclusion.

8.2. CONSTRUCTION THEOREM. If

- (i) α and β are integers with $2 \leq \beta \leq \alpha$,
- (ii) Q is a set of β equally spaced non-negative integers which do not exceed $\alpha - 1$,
- (iii) a is the smallest integer in Q and b is the largest,
- (iv) A is the set of numbers of the form

$$\sum_{j=1}^{\infty} c_j \alpha^{-j},$$

where $c_j \in Q$ for each $j \geq 1$,

$$(v) \quad \Lambda = \left(\frac{b-a}{\alpha-1} \right)^{-\log \beta / \log \alpha},$$

(vi) φ is the function on $[0, \infty)$ to $[0, \infty)$ such that

$$\varphi(x) = \Lambda x^{\log \beta / \log \alpha} \quad (0 \leq x < \infty),$$

(vii) f is the function on R such that $f(x) = \varphi^*((-\infty, x] \cdot A)$,

(viii) Φ is the convex modulus of f ,

$$(ix) \quad B = \sum_{k=1}^{\infty} \left\{ \frac{b-a}{\alpha^k - \alpha^{k-1}} \right\},$$

then

- (1) A is φ -Cantorian,
- (2) $\varphi^*(A) = 1$,
- (3) The Hausdorff dimension of A is $\log \beta / \log \alpha$,
- (4) $\Phi = \langle \varphi, B \rangle$,
- (5) f satisfies a Lipschitz condition of order α and modulus Λ ,

$$(6) \quad f\left(\sum_{j=1}^{\infty} c_j \alpha^{-j}\right) = \frac{\beta-1}{b-a} \sum_{j=1}^{\infty} \frac{c_j - a}{\beta^j},$$

where $c_j \in Q$ for $j = 1, 2, 3, \dots$.

Proof. We divide the proof into parts.

Part I. Construction. Clearly $\varphi \in C$. We take $I = [a(\alpha-1)^{-1}, b(\alpha-1)^{-1}]$ and, letting $k(n) = \beta$ for each n , we employ the method of 6.11 to obtain a set

E. We shall now show that this set E is the set A . Once this has been demonstrated, (1) and (2) will be evident (cf. 6.11), and (3) and (5) will follow from 8.1. Throughout this proof we shall make use of the terminology and notation of 6.11.

Part II. If $J \in H_n$, $|J| = \mu_n$, where

$$\mu_n = \frac{b-a}{\alpha-1} \alpha^{1-n} \quad (n \geq 1).$$

Proof. We assume that our assertion holds for $n = k$, then in view of 6.10 and 5.10, we have $|J'| = \mu_{k+1}$ where

$$\beta\varphi(\mu_{k+1}) = \varphi(\mu_k)$$

for each $J' \in H_{k+1}$. This implies the following relations:

$$\beta\Lambda(\mu_{k+1})^{\log \beta / \log \alpha} = \Lambda(\mu_k)^{\log \beta / \log \alpha};$$

$$\log \beta + \frac{\log \beta}{\log \alpha} \log \mu_{k+1} = \frac{\log \beta}{\log \alpha} \log \mu_k;$$

$$\log \mu_{k+1} = \log \mu_k - \log \alpha;$$

$$\mu_{k+1} = \mu_k \alpha^{-1}.$$

By our assumption, we have then

$$\mu_{k+1} = \left(\frac{b-a}{\alpha-1} \alpha^{1-k} \right) \alpha^{-1} = \frac{b-a}{\alpha-1} \alpha^{1-(k+1)}.$$

Since $H_1 = \{I\}$ and $|I| = (b-a)/(\alpha-1)$, our assertion follows by induction.

Part III. If $J = [a_n, b_n] \in H_n$ with

$$a_n = \sum_{j=1}^{n-1} c_j \alpha^{-j} + a \sum_{j=n}^{\infty} \alpha^{-j}$$

where $c_j \in Q$ for $1 \leq j \leq n-1$, then

$$b_n = \sum_{j=1}^{n-1} c_j \alpha^{-j} + b \sum_{j=n}^{\infty} \alpha^{-j}$$

and

$$H_{n+1} * J = \sum_{m=1}^{\beta} \{[a_n^m, b_n^m]\},$$

where

$$a_n^m = \sum_{j=1}^{n-1} c_j \alpha^{-j} + \left[a + \frac{m-1}{\beta-1} (b-a) \right] \alpha^{-n} + a \sum_{j=n+1}^{\infty} \alpha^{-j}$$

and

$$b_n^m = \sum_{j=1}^{n-1} c_j \alpha^{-j} + \left[a + \frac{m-1}{\beta-1} (b-a) \right] \alpha^{-n} + b \sum_{j=n+1}^{\infty} \alpha^{-j}.$$

Proof. By Part II

$$\begin{aligned} b_n &= a_n + \mu_n = \sum_{j=1}^{n-1} c_j \alpha^{-j} + a \sum_{j=n}^{\infty} \alpha^{-j} + (b-a) \frac{\alpha^{1-n}}{\alpha-1} \\ &= \sum_{j=1}^{n-1} c_j \alpha^{-j} + a \sum_{j=n}^{\infty} \alpha^{-j} + (b-a) \sum_{j=n}^{\infty} \alpha^{-j} \\ &= \sum_{j=1}^{n-1} c_j \alpha^{-j} + b \sum_{j=n}^{\infty} \alpha^{-j}. \end{aligned}$$

Recalling 6.11 and 5.10, we see that

$$H_{n+1} * J = \sum_{m=1}^{\beta} \{[a'_m, b'_m]\},$$

where

$$a'_m = a_n + \frac{m-1}{\beta-1} (\mu_n - \mu_{n+1}), \quad b'_m = a'_m + \mu_{n+1}.$$

We replace a_n by its assumed value and μ_n and μ_{n+1} by their values according to Part II to obtain the relations:

$$\begin{aligned} a'_m &= \sum_{j=1}^{n-1} c_j \alpha^{-j} + a \sum_{j=n}^{\infty} \alpha^{-j} + \frac{m-1}{\beta-1} \left[\frac{b-a}{\alpha-1} (\alpha^{1-n} - \alpha^{-n}) \right] \\ &= \sum_{j=1}^{n-1} c_j \alpha^{-j} + a \sum_{j=n}^{\infty} \alpha^{-j} + \frac{m-1}{\beta-1} (b-a) \alpha^{-n} \\ &= \sum_{j=1}^{n-1} c_j \alpha^{-j} + \left[a + \frac{m-1}{\beta-1} (b-a) \right] \alpha^{-n} + a \sum_{j=n+1}^{\infty} \alpha^{-j}, \end{aligned}$$

and

$$b'_m = \sum_{j=1}^{n-1} c_j \alpha^{-j} + \left[a + \frac{m-1}{\beta-1} (b-a) \right] \alpha^{-n} + b \sum_{j=n+1}^{\infty} \alpha^{-j}.$$

This completes the proof of Part III.

Part IV. If the family H_n consists of those and only those intervals which are of the form $[a_n, b_n]$ where

$$a_n = \sum_{j=1}^{n-1} c_j \alpha^{-j} + a \sum_{j=n}^{\infty} \alpha^{-j} \quad \text{with} \quad c_j \in Q \quad (1 \leq j \leq n-1),$$

then H_{n+1} consists of those and only those intervals which are of the form $[a'_m, b'_m]$ where

$$\begin{aligned} a'_m &= \sum_{j=1}^n c_j \alpha^{-j} + a \sum_{j=n+1}^{\infty} \alpha^{-j}, \\ b'_m &= \sum_{j=1}^n c_j \alpha^{-j} + b \sum_{j=n+1}^{\infty} \alpha^{-j} \quad \text{with} \quad c_j \in Q \quad (1 \leq j \leq n). \end{aligned}$$

Proof. By hypothesis, Q is a set of β equally spaced integers with a and b being the least and the greatest respectively. Therefore,

$$Q = \sum_{m=1}^{\beta} \left\{ a + \frac{m-1}{\beta-1} (b-a) \right\}.$$

By virtue of this fact, 6.11, 5.10, and Part III, our assertion is evidently valid.

Part V. For $n \geq 1$, H_n consists of those and only those intervals which are of the form $[a_n, b_n]$ where

$$a_n = \sum_{i=1}^{n-1} c_i \alpha^{-i} + a \sum_{j=n}^{\infty} \alpha^{-j}, \quad b_n = \sum_{i=1}^{n-1} c_i \alpha^{-i} + b \sum_{j=n}^{\infty} \alpha^{-j},$$

with $c_j \in Q$ for $1 \leq j \leq n-1$.

Proof. $H_1 = \{I\}$ where $I = [a(\alpha-1)^{-1}, b(\alpha-1)^{-1}]$. Evidently

$$a(\alpha-1)^{-1} = a \sum_{i=1}^{\infty} \alpha^{-i}, \quad b(\alpha-1)^{-1} = b \sum_{i=1}^{\infty} \alpha^{-i}.$$

Consequently in view of Part IV, Part V follows by induction.

Part VI. $A = E$.

Proof. This result is an immediate consequence of Part V and 6.11.

Part VII. $\Phi = \langle \varphi, B \rangle$.

Proof. By hypothesis

$$B = \sum_{k=1}^{\infty} \left\{ \frac{b-a}{\alpha^k - \alpha^{k-1}} \right\} = \sum_{k=1}^{\infty} \left\{ \frac{b-a}{\alpha-1} \alpha^{1-k} \right\}.$$

Recalling Part II, we have

$$B = \sum_{k=1}^{\infty} \{\mu_k\} = \sum_{J \in H} \{|J|\},$$

where $H = H_1 + H_2 + H_3 + \dots$. Consequently, 6.10 implies the desired result.

Part VIII.
$$f\left(\sum_{i=1}^{\infty} c_i \alpha^{-i}\right) = \frac{\beta-1}{b-a} \sum_{i=1}^{\infty} (c_i - a) \beta^{-i},$$

where $c_j \in Q$ for $j \geq 1$.

Proof. Let

$$x = \sum_{i=1}^{\infty} c_i \alpha^{-i},$$

where $c_j \in Q$ for $j \geq 1$; for $n \geq 0$ let

$$x_n = \sum_{i=1}^n c_i \alpha^{-i} + a \sum_{j=n+1}^{\infty} \alpha^{-j}.$$

Since $c_j \in Q$ for $j \geq 1$, we have

$$c_j = a + (m_j - 1) \frac{b - a}{\beta - 1},$$

where $1 \leq m_j \leq \beta$. Recalling Part IV, we see that for $n \geq 1$ and $0 \leq k \leq n$, x_{n-k} is a left endpoint of some interval in H_{n-k+1} . Moreover, the family

$$H_{n-k+1}^*[x_{n-k-1}, x_{n-k}]$$

consists of $m_{n-k} - 1$ intervals each of length μ_{n-k+1} . Also in view of the structure of the families, we have (where $U_k = H_{n+1-k}^*[x_{n-1-k}, x_{n-k}]$)

$$\begin{aligned} f(x_n) &= \varphi^*\left(A \cdot \left[\frac{a}{\alpha - 1}, x_n\right]\right) \\ &= \sum_{k=0}^{n-1} \left(\sum_{J \in U_k} \varphi(|J|) \right) \\ &= \sum_{k=0}^{n-1} (m_{n-k} - 1) \varphi(\mu_{n-k+1}) \\ &= \sum_{j=1}^n (m_j - 1) \varphi(\mu_{j+1}) \\ &= \sum_{j=1}^n (m_j - 1) \Delta(\mu_{j+1})^{\log \beta / \log \alpha} \\ &= \sum_{j=1}^n (m_j - 1) \left(\frac{b - a}{\alpha - 1} \right)^{-\log \beta / \log \alpha} \left(\frac{b - a}{\alpha - 1} \alpha^{-j} \right)^{\log \beta / \log \alpha} \\ &= \sum_{j=1}^n (m_j - 1) (\alpha^{-j})^{\log \beta / \log \alpha} \\ &= \sum_{j=1}^n (m_j - 1) \beta^{-j} \\ &= \frac{\beta - 1}{b - a} \sum_{j=1}^n (c_j - a) \beta^{-j}. \end{aligned}$$

Since f is a continuous function

$$f(x) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} \frac{\beta - 1}{b - a} \sum_{j=1}^n (c_j - a) \beta^{-j} = \frac{\beta - 1}{b - a} \sum_{j=1}^{\infty} (c_j - a) \beta^{-j}$$

and our proof is complete.

Part IX. Summary. By Part VI, $A = E$ and hence (1) and (2) are evident (cf. remarks following 6.11). Also (3) and (5) follow from 8.1, (4) from Part VII, and (6) from Part VIII. The proof is now complete.

8.3. Remark. Theorem 8.2 implies several known results. The classes of sets and associated functions which can be treated by the method outlined above include all those discussed by Gilman [5]. In particular, if $\alpha = 3$ and

$\beta = 2$, the function f is the classical "Cantor function". Moreover, by virtue of (5), it is easy to obtain the order and modulus of the Lipschitz condition satisfied by this and other closely related functions. This result is both more general and somewhat more precise than those of Gilman and of Hille and Tamarkin [8]. Statement (3) implies a known result concerning the dimension of Cantor's set [7; 172, 1, 2]. We also notice that in view of (4) the convex modulus of the function f is completely determined. It is clear that this can be done for φ -Cantorian functions associated with φ -Cantorian sets whose structure is considerably more complicated than that of those here considered. The restriction that $k(n) = \beta$ for each n , merely provides us with the convenient radix α representation for the sets A and B .

9. Principles of Fubini and Cavalieri. Throughout this section we let

$$\varphi(x) = \left(\frac{9x}{2}\right)^{\log 2 / \log 10} \quad (0 \leq x < \infty).$$

9.1. DEFINITION. For each $A \subset R$ let $\psi(A) = \varphi^*(A)$.

9.2. DEFINITION. $A \circ B = E_{(x,y)}[x \in A, y \in B]$.

9.3. DEFINITION. If $A \subset R$, and $B \subset R$, then

$$\Pi(A, B) = \psi(A)\psi(B),$$

whenever $0 < \psi(A)$ and $0 < \psi(B)$, and

$$\Pi(A, B) = 0,$$

whenever $\psi(A) = 0$ or $\psi(B) = 0$.

9.4. DEFINITION. Ψ is the function on the set of subsets of $R \circ R$ which is defined as follows:

$\Psi(S) = \inf$ of numbers of the form $\sum_{(X,Y) \in F} \Pi(X, Y)$, where F is such a countable family of pairs of ψ -measurable sets that

$$S \subset \sum_{(X,Y) \in F} X \circ Y.$$

Recalling that empty sums are zero and empty infs are infinite, we conclude that Ψ is a measure for which theorems analogous to 3.4, 3.5, 3.6, 3.7, 3.8, and 3.9 hold and such that $\Psi(A \circ B) = \Pi(A, B)$ whenever A and B are measurable subsets of R .

9.5. NOTATION. If $S \subset R \circ R$, then

$$S^i = \sum_{(x,y) \in S} \{x\},$$

$$S^{\leftarrow} = \sum_{(x,y) \in S} \{y\},$$

$$S_x = E_y[(x, y) \in S],$$

$$S_y = E_x[(x, y) \in S].$$

9.6. DEFINITION.

$$A = E_x[x = \sum_{j=1}^{\infty} c_j 10^{-j}, \text{ where } c_j = 0 \text{ or } 2 \text{ for each } j],$$

$$P = A \circ A,$$

$$P' = E_{(x, y)}[x = \frac{1}{2}y \in A, y \in A],$$

$$P'' = E_{(x, y)}[x = 2 \cdot 5^{-\frac{1}{2}}(x' - \frac{1}{2}y'), y = 2 \cdot 5^{-\frac{1}{2}}(\frac{1}{2}x' + y'), \text{ where } x' \in A, y' \in A].$$

Evidently (see 8.2) we have $\varphi^*(A) = 1$ and also the following:

9.7. THEOREM. $\Psi(P) = 1$.

9.8. LEMMA. *If*

$$u = \sum_{j=1}^{\infty} a_j 10^{-j},$$

where $a_j = 0, \pm 1, \pm 2$ for each j and

$$u = \sum_{j=1}^{\infty} b_j 10^{-j},$$

where $b_j = 0, \pm 1, \pm 2$, for each j , then $a_j = b_j$ for each j .

Proof. Suppose that $a_j = b_j$ for $j < k$ and $a_k - b_k \geq 1$, then

$$\begin{aligned} 0 = u - u &= \sum_{j=1}^{\infty} (a_j - b_j) 10^{-j} \\ &= (a_k - b_k) 10^{-k} + \sum_{j=k+1}^{\infty} (a_j - b_j) 10^{-j} \\ &\geq 10^{-k} - 4 \sum_{j=k+1}^{\infty} 10^{-j} \\ &= 10^{-k} - \frac{4 \cdot 10^{-k}}{9} = \frac{5 \cdot 10^{-k}}{9} > 0. \end{aligned}$$

This is clearly impossible.

9.9. LEMMA. *If x, y, z , and w are elements of A with $x - y = (z - w)/2$, then $x = y$ and $z = w$.*

Proof. Evidently we may write

$$x - y = \sum_{j=1}^{\infty} a_j 10^{-j} \quad (\text{where } a_j = 0, \pm 2 \text{ for each } j)$$

and

$$\frac{1}{2}(z - w) = \sum_{i=1}^{\infty} b_i 10^{-i} \quad (\text{where } b_i = 0, \pm 1 \text{ for each } j).$$

But 9.8 implies $a_i = b_i$ for each j , and hence we conclude that $a_i = b_i = 0$ for each j .

9.10. THEOREM. *For each x , $\psi(P'_x) = 0$ and for each $y \in A$, $\psi(P'_y) = 1$ while $\psi(P'_y) = 0$ for $y \in A^c$. Moreover, $\Psi(P') = \infty$. In fact P is inexpressible as a countable sum of sets of finite Ψ -measure.*

Proof. Suppose $y_1 \in P'_x$ and $y_2 \in P'_x$. We have then

$$x - \frac{1}{2}y_1 = z_1,$$

and

$$x - \frac{1}{2}y_2 = z_2,$$

where y_1, y_2, z_1 , and z_2 are in A . Subtracting these relations, we have

$$\frac{1}{2}(y_2 - y_1) = z_1 - z_2,$$

but by 9.9 this relation cannot hold unless $y_2 = y_1$. We conclude, therefore, that P'_x consists of at most one point. It follows that $\psi(P'_x) = 0$ for each x .

Recalling 9.6, we see that if $y \in A$, then P'_y is a translate of A , and, therefore, $\psi(P'_y) = \psi(A) = 1$. If $y \in A^c$, then P'_y is empty and $\psi(P'_y) = 0$.

The remainder of our theorem now follows from the Fubini theorem. This can also be demonstrated by a direct argument almost identical to the one which will be given in the proof of 9.11.

9.11. THEOREM. *For each x , $\psi(P''_x) = 0$, and for each y , $\psi(P''_y) = 0$. Moreover, $\Psi(P'') = \infty$. In fact, P'' is inexpressible as a countable sum of sets of finite Ψ -measure.*

Proof. We divide the proof into parts.

Part I. For each x , $\psi(P''_x) = 0$.

Proof. If $y_1 \in P''_x$ and $y_2 \in P''_x$ we have the relations

$$(1) \quad x = 2 \cdot 5^{-\frac{1}{2}}(x'_1 - \frac{1}{2}y'_1),$$

$$(2) \quad x = 2 \cdot 5^{-\frac{1}{2}}(x'_2 - \frac{1}{2}y'_2),$$

$$(3) \quad y_1 = 2 \cdot 5^{-\frac{1}{2}}(x'_1 - \frac{1}{2}y'_1),$$

$$(4) \quad y_2 = 2 \cdot 5^{-\frac{1}{2}}(x'_2 - \frac{1}{2}y'_2),$$

where x'_1, y'_1, x'_2 , and y'_2 are in A . (1) and (2) imply

$$x'_1 - x'_2 = \frac{1}{2}(y'_1 - y'_2),$$

but in view of 9.9, this implies $x'_1 = x'_2$ and $y'_1 = y'_2$. It follows then from (3) and (4) that $y_1 = y_2$. We have shown, therefore, that P''_x contains at most one point and hence $\psi(P''_x) = 0$ for each x .

Part II. For each y , $\psi(P''_y) = 0$.

Proof. It is evident that the argument used in proving Part I can easily be modified to give the result desired here.

Part III. $\Psi(P'') = \infty$ and P'' is inexpressible as a countable sum of sets of finite Ψ -measure.

Proof. (By contradiction.) We suppose that P'' is expressible as a countable sum of sets of finite Ψ -measure and let F be such a countable family of ordered pairs of Ψ -measurable sets that

$$P'' \subset \sum_{(X, Y) \in F} X \circ Y$$

with $\Pi(X, Y) < \infty$ for each (X, Y) in F . For $x \in A$ let

$$\bar{x} = E_{(\xi, \eta)} [\xi = 2 \cdot 5^{-\frac{1}{2}}(x - \frac{1}{2}y), \eta = 2 \cdot 5^{-\frac{1}{2}}(\frac{1}{2}x + y) \text{ for some } y \in A],$$

that is, $\bar{x}$ is the rotate of those points of P which are of the form (x, y) where: $y \in A$. Let $(X_0, Y_0) \in F$ and $B = X_0 \circ Y_0$. We notice that for each $x \in A$

$$\psi(\bar{x}^\perp) = 5^{-\frac{1}{2} \log 2 / \log 10}$$

and

$$\psi(\bar{x}^+) = (2 \cdot 5^{-\frac{1}{2}})^{\log 2 / \log 10}.$$

Now if $\psi([\bar{x} \cdot B]^\perp) > 0$ for some $x \in A$, then for the same x , $\psi([\bar{x} \cdot B]^-) > 0$ and since $\Pi(X_0, Y_0) < \infty$ we see that

$$(1) \quad 0 < \psi(X_0) < \infty.$$

In view of the proof of Part I, we obtain the relation

$$(2) \quad \sum_{x \in A} [\bar{x} \cdot B]^\perp \subset X_0,$$

where the summands on the left are disjoint closed sets. It follows from (1) and (2) that

$$E_x[x \in A, \psi([\bar{x} \cdot B]^\perp) > 0]$$

is a countable set. Thus since F is a countable family and A has the power of the continuum, there is $x_0 \in A$ for which

$$\begin{aligned} 0 &= \psi\left(\sum_{(X, Y) \in F} [\bar{x}_0 \cdot (X \circ Y)]^\perp\right) = \psi([\bar{x}_0 \cdot \sum_{(X, Y) \in F} X \circ Y]^\perp) \\ &\geq \psi([\bar{x}_0 \cdot P'']^\perp) = \psi([\bar{x}_0]^\perp) = 5^{-\frac{1}{2} \log 2 / \log 10} > 0. \end{aligned}$$

REFERENCES

1. E. BEST, *A theorem on Hausdorff measure*, Quarterly Journal of Mathematics, vol. 11(1940), pp. 241-248.
2. E. BEST, *On sets of fractional dimensions, III*, Proceedings of the London Mathematical Society (2), vol. 47(1942), pp. 436-454.
3. C. CARATHÉODORY, *Ueber das lineare Mass von Punktmengen, eine Verallgemeinerung des Längenbegriffs*, Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen (1914), pp. 404-426.
4. C. CARATHÉODORY, *Vorlesungen über Reale Funktionen*, Leipzig, 1918.
5. R. E. GILMAN, *A class of functions continuous but not absolutely continuous*, Annals of Mathematics, vol. 33 (1932), pp. 433-442.
6. H. HAHN, *Theorie der reellen Funktionen*, vol. I, Berlin, 1921.
7. F. HAUSDORFF, *Dimension und äusseres Mass*, Mathematische Annalen, vol. 79(1919), pp. 157-179.
8. E. HILLE AND J. D. TAMARKIN, *Remarks on a known example of a monotone continuous function*, American Mathematical Monthly, vol. 36(1929), pp. 255-264.
9. S. SAKS, *Theory of the Integral*, Warsaw, 1937.
10. G. VITALE AND G. SANSONE, *Moderna teoria delle funzioni di variabile reale*, vol. I, Bologna, 1935.

UNIVERSITY OF NEVADA AND UNIVERSITY OF CALIFORNIA.

